Modeling Continuous Time Series Driven by Fractional Gaussian Noise

By

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Abstract:

We consider the stochastic differential equations, $dX(t) = \theta X(t)dt + dB_H(t)$; $t > 0$, and $dX(t) = \theta(t)X(t)dt + dB_H(t)$; $t > 0$ where $B_H(t)$ is fractional Brownian motion. We find solutions for these differential equations and show the existence of the integrals related to these solutions. We then show that $B_H(t)$ is not a martingale. This implies that several conventional methods for defining integrals on fractional Brownian motion are inadequate. We demonstrate the existence of an estimator for θ or $\theta(t)$ which depends on the existence of integrals of certain integrals with respect to fractional Brownian motion. We conclude by showing the existence and Riemann sum approximations for these integrals.

1. Introduction

In this paper, we demonstrate the existence of optimal statistical estimators for parameters of certain forms of stochastic differential equations driven by fractional Gaussian noise. Dobrushin (1979) and Major (1981) both consider linear and nonlinear functionals of self-similar Gaussian fields with stationary increments. Fractional Brownian motion is such a process. This type of random noise appears in certain physical processes that exhibit correlations that decrease slowly with time and low frequency power. Some physical processes possess the fractal property of self-similarity, which is a basic property of fractional Brownian motion. Previously established parametric estimators mainly deal with random noise in the form of Gaussian white noise and its standard Brownian motion, although algorithms have also been derived to handle random processes in the form of square-integrable martingales, which generalize the Brownian motion noise process. Both

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man-made and natural processes appear to exhibit randomness in the form of fractional Brownian motion or fractional Gaussian noise.

The fractal property of statistical self-similarity often appears in geophysical processes. In geology and hydrology, models with fractional random processes prove useful. River discharges tend to exhibit clusters of high periods and low periods and thus exhibit long-term dependencies (Mandelbrot, 1983). Gregotski, Jensen, and Arkani-Hamed (1991) demonstrate experimental data indicating that spatial magnetic patterns of certain geographical locations behave in a statistical self-similar way where the independent variables are spatial processes. Self-similarity also is modeled for communication channels and internet communication. Random errors in communication channels may occur in groups of bursts, where this groups of bursts are themselves grouped in bursts (Barton and Poor, 1988). Stewart et al. (1993) show that radar images from natural "clutter sources" have a texture that looks like fractional Brownian motion in two dimensions with the independent variables being distances. Finally, we note that Wegman and Habib (1992) apply the class of stochastic differential equation models we describe here to describe sub-threshold neuron-firing processes.

2. Solution of the Stochastic Differential Equations

We consider first the parametric model as the stochastic differential equation

$$dX(t) = \theta X(t)dt + dB_H(t);\ t > 0. \tag{2.1}$$

$B_H(t)$ is fractional Brownian motion. Let $\{B(t)\colon t \in R\}$ be a standard Brownian motion process, then fractional Brownian motion, B_H for given $H \in (1/2, 1)$ is defined as follows:

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0} (\,|\,t-\tau\,|^{\,H-1/2} - \,|\,\tau\,|^{\,H-1/2})dB(\tau) + \int_{0}^{t} |\,t-\tau\,|^{\,H-1/2}dB(\tau)\right\}. \tag{2.2}$$

Notice for $H = 1/2$, fractional Brownian motion coincides with ordinary Brownian motion.

To develop the solution to (2.1), first of all, consider the homogeneous form of this differential equation $dX(t) = \theta X(t)dt$. It is straightforward to see that $X(t) = e^{\theta t}X(0)$ is the homogeneous solution. Assume, then, that the particular solution has the form $X(t) = e^{\theta t}Y(t)$. Under this assumption we have the following differential equation

$$dY(t) = e^{-\theta t}dB_H(t).$$

This equation is formally equivalent to the integral equation

$$Y(t) = \int_0^t e^{-\theta\tau}dB_H(\tau).$$

Substituting this solution for $Y(t)$ back into the original yields the particular solution

$$X(t) = e^{\theta t}\int_0^t e^{-\theta\tau} dB_H(\tau).$$

Thus formally the general solution is

$$X(t) = e^{\theta t}X(0) + e^{\theta t}\int_0^t e^{-\theta\tau} dB_H(\tau). \tag{2.3}$$

For the case where $X(0)$ is zero the solution is

$$X(t) = e^{\theta t}\int_0^t e^{-\theta\tau} dB_H(\tau). \tag{2.4}$$

These are formal solutions to the stochastic differential equation (2.1) since existence of the stochastic integrals in equations (2.3) and (2.4) have not been established. In the case of an equation driven by a martingale, the existence of these integrals has been established. However, as we shall shortly see, fractional Brownian motion is not a martingale, hence, we need to establish the existence of these integrals separately.

These solutions can easily be generalized to a nonparametric form, where the θ term is an unknown function rather than an unknown constant

$$dX(t) = \theta(t)X(t)dt + dB_H(t). \tag{2.5}$$

Now consider a solution to the homogeneous differential equation of the form $X(t) = A(t)X(0)$. It is straightforward to show that

$$A(t) = e^{\int_0^t \theta(\tau)d\tau}.$$

Hence, substituting for $A(t)$, the homogeneous solution is as follows:

$$X(t) = e^{\int_0^t \theta(\tau)d\tau} X(0).$$

Now, assuming a particular solution to be of the form

$$X(t) = A(t)Y(t)$$

where $A(t)$ is as before and $Y(t)$ is an unknown process, we find

$$X(t) = A(t)\int_0^t (1/A(\tau))dB_H(\tau).$$

So the general solution is

$$X(t) = e^{\int_0^t \theta(\alpha)d\alpha} X(0) + e^{\int_0^t \theta(\alpha)d\alpha} \int_0^t e^{-\int_0^\tau \theta(\alpha)d\alpha} dB_H(\tau),$$

or assuming $X(0) = 0$,

$$X(t) = e^{\int_0^t \theta(\alpha)d\alpha} \int_0^t e^{-\int_0^\tau \theta(\alpha)d\alpha} dB_H(\tau)$$

or equivalently

$$X(t) = \int_0^t e^{\int_\tau^t \theta(\alpha)d\alpha} dB_H(\tau). \tag{2.6}$$

As before, these are formal manipulations since we have not yet proved the existence of the integrals involved. As mentioned above if B_H were a martingale, the existence of integrals in expressions (2.3), (2.4), and (2.6) would be demonstrated. However, B_H is not a martingale, and hence we need to appeal to first principles in order to demonstrate the existence of these integrals. We base the result on the following theorem.

Theorem 2.1 (Cramer and Leadbetter, 1967, p. 90): If the covariance function $R(s,r)$ of X is of bounded variation in $[0,t] \times [0,t]$ and f is a deterministic function f is such that $\int_0^t \int_0^t f(s)f(r)d_{s,r}R(s,r)$ exists as a Riemann-Stieltjes integral, then $\int_0^t f(s)dX(s)$ is well defined.

The covariance of fractional Brownian motion is given by

$$R_{B_H}(s, t) = \frac{V_H}{2}(| s | ^{2H} + | t | ^{2H} - | t - s | ^{2H}) \tag{2.7}$$

where $V_H = var[B_H(1)] = \frac{-\Gamma(2-2H) cos(\pi H)}{\pi H (2H-1)}$ such that $H \in (1/2, 1)$ (Barton and Poor, 1988). For $H > 1/2$, this $R_{B_H}(s, t)$ is clearly of bounded variation so that by Theorem 2.1, the integrals in (2.3), (2.4), and (2.6) exist and are well-defined.

3. B_H is not a Martingale

As we have just indicated, integrals of a continuous process with respect to B_H are well defined if B_H is a square-integrable martingale or a local square-integrable martingale. Unfortunately, this not the case will be seen in the theorems to follow. Although a martingale is a local martingale, what follows first is a proof that fractional Brownian motion is not a martingale, which can be easily generalized to show that B_H is also not a local martingale.

Theorem 3.1: Let $\{B_H(t)\colon -\infty < t < \infty\}$ be a fractional Brownian motion. Let the σ-algebra filtration $\{\mathcal{A}_t\colon -\infty < t < \infty\}$ be the filtration to which a Brownian motion B is adapted and, let B_H be derived from B. $\{B_H(t), A_t\colon t \geq 0\}$ is not a martingale.

Proof: Let $t > s \geq 0$.

$$E[B_H(t) \mid \mathcal{A}(s)]$$

$$= \frac{1}{\Gamma(H+1/2)}\{E(\int_{-\infty}^{0}(\mid t-\tau\mid^{H-1/2} - \mid\tau\mid^{H-1/2})dB(\tau) \mid \mathcal{A}(s)) + E(\int_{0}^{t}\mid t-\tau\mid^{H-1/2}dB(\tau) \mid \mathcal{A}(s))\}$$

$$= \frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{0}(\mid t-\tau\mid^{H-1/2} - \mid\tau\mid^{H-1/2})dB(\tau)$$

$$+ \int_{0}^{s}\mid t-\tau\mid^{H-1/2}E[dB(\tau) \mid \mathcal{A}(s)] + \int_{s}^{t}\mid t-\tau\mid^{H-1/2}E[dB(\tau) \mid \mathcal{A}(s)]. \qquad (3.1)$$

Since $B(\tau)$ has independent increments, $E[dB(\tau) \mid \mathcal{A}(s)] = 0$ for all $\tau \geq s$. Hence, the last term on the right-side of the equation (3.1) equals 0, and therefore we have as follows:

$$\frac{1}{\Gamma(H+1/2)}E\left(\int_{-\infty}^{0}(\mid t-\tau\mid^{H-1/2} - \mid\tau\mid^{H-1/2})\,dB(\tau) + \int_{0}^{t}\mid t-\tau\mid^{H-1/2}dB(\tau)) \mid \mathcal{A}(s)\right)$$

$$= \frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{0}(\mid t-\tau\mid^{H-1/2} - \mid\tau\mid^{H-1/2})\,dB(\tau) + \int_{0}^{s}\mid t-\tau\mid^{H-1/2}dB(\tau).$$

Notice that the right-hand side of the above expression depends explicitly on t; this is not equal to $B_H(s)$ since $B_H(s)$ is

$$B_H(s) = \frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{0}(\mid s-\tau\mid^{H-1/2} - \mid\tau\mid^{H-1/2})\,dB(\tau) + \int_{0}^{s}\mid s-\tau\mid^{H-1/2}dB(\tau).$$

Hence, $\{B_H(t),\mathcal{A}(t)\colon t \geq 0\}$ is not a martingale, and the theorem is proved.

Corollary 3.2: Let $\{B_H(t)\colon -\infty < t < \infty\}$ be a fractional Brownian motion. Let the σ-algebra filtration $\{\mathcal{A}_t\colon -\infty < t < \infty\}$ be the filtration to which a Brownian motion B is adapted and, let B_H be derived from B. $\{B_H(t), A_t\colon t \geq 0\}$ is not a local martingale.

Proof: Suppose $\{B_H(t),\mathcal{A}(t)\colon t \geq 0\}$ is a local martingale. There exists $\{T_n\}$ a sequence of stopping times such that $T_n \to \infty$ and $T_n \leq T_{n+1}$. Then $B_H(T_n \wedge t)I_{(T_n>0)}$, where the I function is an indicator function, is a martingale. By the definition of a martingale the following must then hold:

$$E\left[B_H(T_n \wedge t)I_{(T_n>0)} \mid \mathcal{A}(s)\right] = B_H(T_n \wedge s)I_{(T_n>0)}B_H(T_n \wedge s)I_{(T_n>0)}$$

$$= \frac{1}{\Gamma(H+1/2)} I_{(T_n>0)} \Big\{ \int_{-\infty}^{0} |\ T_n \wedge s - \tau\ |^{\ H-1/2} - \ |\ \tau\ |^{\ H-1/2}) dB(\tau) +$$

$$\int_{-\infty}^{T_n \wedge s} |\ T_n \wedge s - \tau\ |^{\ H-1/2} dB(\tau) \Big\}.$$

Let *f* be defined such that

$$f(r,\alpha) = I_{(-\infty,0)}(\ |\ r - \alpha\ |^{\ H-1/2} - \ |\ \alpha\ |^{\ H-1/2}) + I_{[0,r)}\ |\ r - \alpha\ |^{\ H-1/2}.$$

Using this definition of *f* to simplify formulas,

$$B_H(T_n \wedge s) I_{(T_n>0)} = I_{(T_n>0)} \int_{-\infty}^{T_n \wedge s} f(T_n \wedge s,\tau) dB(\tau)$$

or

$$B_H(T_n \wedge s) = \int_{0}^{T_n \wedge s} f(T_n \wedge s,\tau) dB(\tau).$$

Hence, if $B_H(T_n \wedge t) I_{(T_n>0)}$ is assumed to be a martingale, then the following relationship has been shown to be true:

$$E[B_H(T_n \wedge t) I_{(T_n>0)} \mid \mathcal{A}(s)] = I_{(T_n>0)} \int_{-\infty}^{T_n \wedge s} f(T_n \wedge s,\tau) dB(\tau), \textit{ for } t \geq s$$

$$= \int_{0}^{T_n \wedge s} f(T_n \wedge s,\tau) dB(\tau).$$

Letting Ω be the sample space, by the definitions of the expected value and the indicator function,

$$E[B_H(T_n \wedge t) I_{(T_n>0)}] = \int_{\Omega} \int_{0}^{T_n \wedge t} f(t \wedge T_n,\tau) dB(\tau,\omega) dP(\omega)$$

where P is the probability measure and $\omega \in \Omega$. By the measure theoretic definition of conditional expected value, given $A \in \mathcal{A}(s)$,

$$\int_{A} E[B_H(T_n \wedge t) I_{(T_n>0)} \mid \mathcal{A}(s)] dP(\omega) = \int_{A} \int_{0}^{T_n \wedge t} f(T_n \wedge t,\tau) dB(\tau,\omega) dP(\omega)$$

so that

$$\int_{A} \int_{0}^{T_n \wedge s} f(T_n \wedge s,\tau) dB(\tau,\omega) dP(\omega) = \int_{A} \int_{0}^{T_n \wedge t} f(T_n \wedge t,\tau) dB(\tau,\omega) dP(\omega), \textit{ for } t \geq s.$$

However, this cannot be true since t is not included in the deterministic function f of the integral on the left-hand side of the last equation. Therefore, we have a contradiction and $\{B_H(t),\mathcal{A}(t)\colon t \geq 0\}$ must not be a local martingale, and the theorem is proved.

Using the equation for $E[B_H(T_n \wedge t) \mid \mathcal{A}(s)]$ in the proof that fractional Brownian motion is not a local martingale, we can generalize one step further and claim that B_H is not a semimartingale. In proving that B_H is not a semimartingale, the following result is needed:

Theorem 3.3 (Shiryayev, 1984, p 213): If W and Y are to random variables such that $W \leq Y$ a.s., then

$$E[W \mid \mathcal{A}] \leq E[Y \mid \mathcal{A}] \text{ a.s.}$$

Now the theorem claiming that fractional Brownian motion is not a semimartingale along with its proof will be given.

Corollary 3.4: $B_H = \{B_H(t)\colon t \in (-\infty, \infty)\}$ is not a semimartingale.

Proof: Suppose B_H is a semimartingale. Then

$$B_H(t) = B_H(0) + M(t) + A(t)\ t \geq 0 \text{ a.s.}$$

or

$$B_H(t) = M(t) + A(t) \text{ since } B_H(0) = 0$$

where M is a local martingale and A is a right-continuous adapted process with locally bounded variation sample paths. Thus,

$$B_H(t) - A(t) = M(t),\ t \geq 0$$

is a local martingale. So there exists an increasing stopping time sequence $\{T_n\}$ such that $T_n \to \infty$ as $n \to \infty$ and $B_H(T_n \wedge t) - A(T_n \wedge t)$ is a martingale. Given the adapting σ-algebra $\mathcal{A} = \{\mathcal{A}(t)\colon t \geq 0\}$ and using the definition of a martingale,

$$E\{B_H(T_n \wedge t) \mid \mathcal{A}(s)] - E[A(T_n \wedge t) \mid \mathcal{A}(s)] = B_H(T_n \wedge s) - A(T_n \wedge s) \text{ for all } s < t.$$

But $B_H(r)$ is $\int_{-\infty}^{r} f(r,\alpha)dB(\alpha)$ where

$$f(r,\alpha) = I_{(-\infty,0)}(\,|\,r-\alpha\,|^{\,H-1/2} - \,|\,\alpha\,|^{\,H-1/2}) + I_{[0,r)}\,|\,r-\alpha\,|^{\,H-1/2}.$$

Substituting the definition of B_H, using f for the needed integrand, and substituting the expression for $E\{B_H(T_n \wedge t) \mid \mathcal{A}(s)\}$ as given in the proof that B_H is not a local martingale, we have:

$$E\{B_H(T_n \wedge t) \mid \mathcal{A}(s)] - E[A(T_n \wedge t) \mid \mathcal{A}(s)] = B_H(T_n \wedge s) - A(T_n \wedge s) \text{ for all } s < t \ \Rightarrow$$

$$\int_{-\infty}^{T_n \wedge s} f(T_n \wedge t, \alpha) dB(\alpha) - E[A(T_n \wedge t) \mid \mathcal{A}(s)]$$

$$= \int_{-\infty}^{T_n \wedge s} f(T_n \wedge s, \alpha) dB(\alpha) - A(T_n \wedge s) \ \Rightarrow$$

$$E[A(T_n \wedge t) \mid \mathcal{A}(s)] - A(T_n \wedge s) = \int_{-\infty}^{T_n \wedge s} f(T_n \wedge t, \alpha) dB(\alpha) - \int_{-\infty}^{T_n \wedge s} f(T_n \wedge s, \alpha) dB(\alpha).$$

Since A is of locally bounded variation, on every finite interval, it must be the difference of two monotonic functions. This implies that $E[A(T_n \wedge t) \mid \mathcal{A}(s)]$ must also be the difference of two monotone functions for $s \in [0,\ t]$ by the theorem that immediately preceded this present result. This means that $E[A(T_n \wedge t) \mid \mathcal{A}(s)]$ must also be of locally bounded variation, and so $E[A(T_n \wedge t) \mid \mathcal{A}(s)] - A(T_n \wedge s)$ must be of locally bounded variation.

Since B is almost surely not differentiable for all $t \in (-\infty, \infty)$, it is not of bounded variation for all intervals. This implies by definition that for all $r \in (-\infty, \infty)$,

$$\int_{0}^{T_n \wedge r} \mid dB(\alpha) \mid \ = \infty \text{ (Shiryayev, 1981, p. 201).}$$

But

$$\min_{\alpha \in (0, T_n \wedge s)} [(T_n \wedge t - \alpha)^{H-1/2} - (T_n \wedge s - \alpha)^{H-1/2}]$$

$$= (T_n \wedge t)^{H-1/2} - (T_n \wedge s)^{H-1/2} = (T_n \wedge t)^{H-1/2} - \mathrm{s}^{H-1/2} > 0$$

for the case where the random process $T_n > s$. There is no loss of generality in the arguments to follow by assuming the special case, for which $T_n > s$, since in order for B_H to be a semimartingale, the arguments must not lead to a contradiction under any circumstance. Now

$$\int_{0}^{T_n \wedge s} \mid (T_n \wedge t)^{H-1/2} - s^{H-1/2} dB(r) \mid \ =$$

$$(T_n \wedge t)^{H-1/2} - s^{H-1/2} \int_{0}^{T_n \wedge s} \mid dB(r) \mid \ = \infty.$$

This implies

$$\int_{0}^{T_n \wedge s} \mid d_r \int_{0}^{T_n \wedge r} \left\{ (T_n \wedge t)^{H-1/2} - s^{H-1/2} \right\} dB(\alpha) \mid \ = \infty,$$

where d_r is the differential with respect to r symbol. In other words, this last equation states that the limiting sum of the variations of the random process, $\int_0^{T_n \wedge r} \left\{ (T_n \wedge t - s)^{H-1/2} \right\} dB(\alpha)$, is unbounded. Since

$$[(T_n \wedge t - \alpha)^{H-1/2} - (T_n \wedge s - \alpha)^{H-1/2}] \geq (T_n \wedge t - s)^{H-1/2},$$

the limiting sum of the variations of the stochastic process represented by $\int_0^{T_n \wedge r} \left\{ (T_n \wedge t - \alpha)^{H-1/2} - (T_n \wedge s - \alpha)^{H-1/2} \right\} dB(\alpha)$ must also be unbounded. Moreover, this means that the random process

$$\int_{-\infty}^{T_n \wedge r} [f(T_n \wedge t, \alpha) - f(T_n \wedge s, \alpha)] dB(\alpha) =$$
$$\int_{-\infty}^{0} [f(T_n \wedge t, \alpha) - f(T_n \wedge s, \alpha)] dB(\alpha)$$
$$+ \int_0^{T_n \wedge r} \left\{ (T_n \wedge t - \alpha)^{H-1/2} - (T_n \wedge s - \alpha)^{H-1/2} \right\} dB(\alpha)$$

must also be of unbounded variation in the interval $[0, T_n \wedge s]$, that is,

$$\int_0^{T_n \wedge s} \left| d_r \int_{-\infty}^{T_n \wedge r} [f(T_n \wedge t, \alpha) - f(T_n \wedge s, \alpha)] dB(\alpha) \right| = \infty.$$

In other words, $\int_{-\infty}^{T_n \wedge s} f(T_n \wedge t,\ \alpha) dB(\alpha) - \int_{-\infty}^{T_n \wedge s} f(T_n \wedge s,\ \alpha) dB(\alpha)$ is not of locally bounded variation. This is a contradiction to the fact that this process was set equal to $E[A(T_n \wedge t) \mid \mathcal{A}(s)] - A(T_n \wedge s)$, which was shown to be of locally bounded variation. Hence, B_H must not be a semimartingale, and the theorem is proved.

4. Christopeit's Quasi-Least-Squares Methods and Its Implications

Given the fractional Brownian motion process B_H for $H \in (1/2,\ 1)$, we now consider the estimation problem for parametric model given by

$$dX(t) = \theta X(t) dt + dB_H(t)$$

by first considering a continuous extension of a least squares method. The integral form of the model fits the stochastic process regression model as given in Christopeit (1986), except for the fact that the noise, which is fractional Brownian motion here, is not a martingale.

Christopeit's model is represented by

$$Y(t) = Y(0) + \theta \int_0^t X(s) dF(s) + M(t)$$

where F is an increasing process and M is a martingale. The quasi-least-squares estimate of θ as given in Christopeit is as follows:

$$\hat{\theta}(t) = [\int_0^t X^2(s) dF(s)]^{-1} \int_0^t X(s) dY(s)$$

for the sample path in $[0, t]$. This method is called quasi-least squares because given a discrete partition of the time interval involved a least-squares estimate converges to the above estimate.

Although B_H is not a martingale, the quasi-least-squares estimator as given for the model that we are considering is given by

$$\hat{\theta} = \int_0^t X(s) dX(s) / \int_0^t X^2(s) ds.$$

The integral in the numerator will be shown to be well defined in what follows.

The fact that the noise, being fractional Brownian motion, is not a martingale only affects the asymptotic properties and not the fact that the estimator is a quasi-least-squares estimate as long as the integrals in the estimator are well defined. Thus, the above estimator may still be a legitimate quasi-least-squares estimator although its asymptotic properties may not be as desirable. But the existence of integral in the numerator, $\int_0^t X(s) dX(s)$, must be demonstrated when the noise is not a martingale.

In order to determine whether $\int_0^t X(s) dX(s)$ exists as well as to decompose the estimator into the sum of the true value of the parameter, θ, and an error term, note that the estimator derived above can be formally represented by

$$\hat{\theta} = \int_0^t X(s)[\theta X(s) ds + dB_H(s)] / \int_0^t X^2(s) ds$$

or equivalently

$$\hat{\theta} = \theta + \left\{ \int_0^t X(s) dB_H(s) / \int_0^t X^2(s) ds \right\}.$$

This means that $\int_0^t X(s)dX(s)$ may be defined in terms of $\int_0^t X^2(s)ds$ and $\int_0^t X(s)dB_H(s)$ where θ is the true parameter value. The first integral $\int_0^t X^2(s)ds$ can be interpreted as either a quadratic mean integral or a sample path (Lebesgue or Riemann) integral, and it is finite since $X^2(s)$ is bounded almost surely in $[0, t]$. This is also why the denominator of the estimator, which is this same integral, is not of concern. The second integral, namely, $\int_0^t X(s)dB_H(s)$, will be shown to exist in the next section.

Since B_H is not a martingale, a local martingale, nor a semimartingale, the integrals $\int_0^t X(s)dB_H(s)$, where $dX(s) = \theta(s)X(s)ds + dB_H(s)$ are not defined in the conventional sense of stochastic integrals defined with respect to martingales or their variants. Thus in order for this estimator to make sense, we must develop a rigorous definition for this type of stochastic integral.

5. Defining the Integrals

First recall from the previous section that given the stochastic differential equation as stated above,

$$X(s) = \int_0^s e^{\int_\tau^s \theta(\alpha)\, d\alpha} dB_H(\tau) \text{ for } X(0) = 0, \tau \geq 0.$$

Thus $\int_0^t X(s)dB_H(s)$ may be defined as

$$\int_0^t X(s)dB_H(s) = \int_0^t \int_0^s e^{\int_\tau^s \theta(\alpha)\, d\alpha} dB_H(\tau)dB_H(s). \tag{5.1}$$

Thus we would like to show the existence of the integral on the right-hand side of (5.1). Define a function ζ represented by

$$\zeta(s,\tau) = e^{\int_\tau^s \theta(\alpha)\, d\alpha}.$$

Partition $[0,t]$ such that $\pi_n = \{0 = v_0, v_1, v_2, \ldots, v_n = t \leq T\}$ for $T \in (-\infty, \infty)$. Define a step function $\zeta_n(s, \tau) = e^{\int_{v_{k-1}}^{v_j} \theta(\alpha)\, d\alpha}$ if $\tau \in [v_{k-1}, v_k)$, $s \in (v_{j-1}, v_j]$, $j, k = 0,1,\ldots,n$ and $\zeta_n(s, \tau) = 0$ if $\tau > s$ or $s > t$. For this step function and analogously for any step function, we define the stochastic integral in the following way:

$$\int_0^t \int_0^s \zeta_n(s,w) dB_H(w) dB_H(s) \equiv \sum_{j=1}^{n} \sum_{k=1}^{j} \zeta_n(v_{j-1},v_{k-1})[B_H(v_k) - B_H(v_{k-1})][B_H(v_j) - B_H(v_{j-1})]. \quad (5.2)$$

where $v_j,\ v_k \in \{v_0 = 0,\ v_1,\ v_2,\ldots,\ v_n = t \leq T\}$. Thus $\zeta_n(s,\ u) \to e^{\int_u^s \theta(\alpha)\, d\alpha}$ if $u \leq s$ and $\zeta_n(s,\ u) \to 0$ if $u > s$. Since ζ_n is uniformly bounded by $max(e^{\int_u^s \theta(\alpha)\, d\alpha})$, for $s,u \in [0,t]$, it converges uniformly in $s,u \in [0,t]$. We now wish to show that the right-hand side of (5.2) converges as the norm of the partition, π_n, approaches 0.

To see this, we will want to show the right-hand side of (5.2) is a Cauchy sequence in quadratic mean. Since the space on which B_H lives is a complete Hilbert space, each Cauchy sequence must converge to a limit. This limit will be by definition the integral. Let us begin by observing the following Theorem.

Theorem 5.1 (Soong, 1973, p. 28 and p. 32): Let $W_1,\ldots,W_4$ be 4 jointly Gaussian zero mean random variables. Then,

$$E[W_1 \ldots W_4] = E[W_1 W_2]E[W_3 W_4] + E[W_1 W_3]E[W_2 W_4] + E[W_1 W_4]E[W_2 W_3].$$

Let π_n and π_m be two partitions of $[0,t]$. Without loss of generality, we may consider the union of these partitions, $\pi_n \cup \pi_m = \pi_{nm} = \{v_1 \leq ... \leq v_N\}$ where $N = m + n$. Let $h = ||\pi_{nm}||$. Some of the v_j's may be redundant. However, the differences, $B_H(v_k) - B_H(v_{k-1})$, in this case will be 0. We have the following result.

Lemma 5.2:

1) $E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})] =$

$$\frac{V_H}{2}[- |v_i - v_j|^{2H} + |v_{i-1} - v_j|^{2H} + |v_i - v_{j-1}|^{2H} - |v_{i-1} - v_{j-1}|^{2H}].$$

2) $E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})][B_H(v_k) - B_H(v_{k-1})][B_H(v_l) - B_H(v_{l-1})]$

$$\leq 3[(2t+1)h]^2 = O(h^2).$$

Proof:

By the Soong Theorem 5.1, since B_H is a Gaussian random variable

$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})][B_H(v_k) - B_H(v_{k-1})][B_H(v_l) - B_H(v_{l-1})] \leq$

$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})]E[B_H(v_k) - B_H(v_{k-1})][B_H(v_l) - B_H(v_{l-1})] +$

$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_k) - B_H(v_{k-1})]E[B_H(v_j) - B_H(v_{j-1})][B_H(v_l) - B_H(v_{l-1})] +$

$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_l) - B_H(v_{l-1})]E[B_H(v_j) - B_H(v_{j-1})][B_H(v_k) - B_H(v_{k-1})].$

Let us consider expressions of the form

$$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})] =$$

$$E[B_H(v_i)B_H(v_j) - B_H(v_{i-1})B_H(v_j) - B_H(v_i)B_H(v_{j-1}) + B_H(v_{i-1})B_H(v_{j-1})]. \tag{5.3}$$

Since B_H is a zero mean Gaussian process, the right-hand side of (5.3) represents four covariances. From equation (2.7) we have

$$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})] =$$

$$\frac{V_H}{2}[|v_i|^{2H} + |v_j|^{2H} - |v_i - v_j|^{2H} - |v_{i-1}|^{2H} - |v_j|^{2H} + |v_{i-1} - v_j|^{2H} - |v_i|^{2H} - |v_{j-1}|^{2H}$$

$$+ |v_i - v_{j-1}|^{2H} + |v_{i-1}|^{2H} + |v_{j-1}|^{2H} - |v_{i-1} - v_{j-1}|^{2H}] =$$

$$\frac{V_H}{2}[- |v_i - v_j|^{2H} + |v_{i-1} - v_j|^{2H} + |v_i - v_{j-1}|^{2H} - |v_{i-1} - v_{j-1}|^{2H}].$$

Let us consider $|v_{i-1} - v_j|^{2H} - |v_i - v_j|^{2H}$ and let us assume for the moment that $v_j > v_i$. Then

$$|v_{i-1} - v_j|^{2H} - |v_i - v_j|^{2H} = (v_j - v_{i-1})^{2H} - (v_j - v_i)^{2H}$$

$$= (v_j - v_i + v_i - v_{i-1})^{2H} - (v_j - v_i)^{2H}$$

$$\leq (v_j - v_i + h)^{2H} - (v_j - v_i)^{2H}$$

$$\leq max\{h^2 + 2h(v_j - v_i),\, h\}$$

$$\leq (2t + 1)h.$$

If $v_{i-1} \leq v_j \leq v_i$, then either $v_j = v_{i-1}$ or $v_j = v_i$ so that

$$|v_{i-1} - v_j|^{2H} - |v_i - v_j|^{2H} \leq (v_i - v_{i-1}) \leq h.$$

If $v_j \leq v_{i-1}$, then as before

$$|v_{i-1} - v_j|^{2H} - |v_i - v_j|^{2H} \leq (2t + 1)h.$$

It follows then that

$$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})] \leq (2t + 1)h.$$

Similarly for the other five combinations, so that

$$E[B_H(v_i) - B_H(v_{i-1})][B_H(v_j) - B_H(v_{j-1})][B_H(v_k) - B_H(v_{k-1})][B_H(v_l) - B_H(v_{l-1})] \leq$$

$3[(2t + 1)h]^2 = O(h^2)$. ∎

We are now in a position to prove the following result.

Lemma 5.3:

$$\sum_{j=1}^{n}\sum_{k=1}^{j}\zeta_n(v_{j-1},v_{k-1})[B_H(v_k)-B_H(v_{k-1})]\,[B_H(v_j)-B_H(v_{j-1})]$$

is a Cauchy sequence in quadratic mean.

Proof: First note that for $a,\ b\in(-\infty,\infty),\ |a-b|^2\le 2|a|^2+|b|^2$. Thus we have

$$E\Big|\sum_{i=1}^{n}\sum_{j=1}^{i}\zeta_n(v_{i-1},v_{j-1})\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_j)-B_H(v_{j-1})\big]-$$
$$\sum_{k=1}^{m}\sum_{l=1}^{k}\zeta_m(v_{k-1},v_{l-1})\big[B_H(v_k)-B_H(v_{k-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]\Big|^2\le$$

$$E\Big\{2\Big|\sum_{i=1}^{n}\sum_{j=1}^{i}\zeta_n(v_{i-1},v_{j-1})\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_j)-B_H(v_{j-1})\big]\Big|^2\Big\}+$$

$$E\Big\{2\Big|\sum_{k=1}^{m}\sum_{l=1}^{k}\zeta_m(v_{k-1},v_{l-1})\big[B_H(v_k)-B_H(v_{k-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]\Big|^2\Big\}=$$

$$2\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{k}\zeta_n(v_{i-1},v_{j-1})\zeta_n(v_{k-1},v_{l-1})E\Big\{\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_j)-B_H(v_{j-1})\big]$$
$$\big[B_H(v_k)-B_H(v_{k-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]\Big\}+$$

$$2\sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{k=1}^{m}\sum_{l=1}^{k}\zeta_m(v_{i-1},v_{j-1})\zeta_m(v_{k-1},v_{l-1})E\Big\{\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_j)-B_H(v_{j-1})\big]$$
$$\big[B_H(v_k)-B_H(v_{k-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]\Big\}. \qquad (5.4)$$

Both terms in the expression (5.4) are similar except for the m and n. Consider the first term.

$$2\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{k}\zeta_n(v_{i-1},v_{j-1})\zeta_n(v_{k-1},v_{l-1})E\Big\{\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_j)-B_H(v_{j-1})\big]$$
$$\big[B_H(v_k)-B_H(v_{k-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]\Big\}=$$

$$2\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{k}\zeta_n(v_{i-1},v_{j-1})\zeta_n(v_{k-1},v_{l-1})$$
$$\Big\{E\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_j)-B_H(v_{j-1})\big]E\big[B_H(v_k)-B_H(v_{k-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]+$$

$$E\big[B_H(v_i)-B_H(v_{i-1})\big]\big[B_H(v_k)-B_H(v_{k-1})\big]E\big[B_H(v_j)-B_H(v_{j-1})\big]\big[B_H(v_l)-B_H(v_{l-1})\big]+$$

$$E\left[B_H(v_i)-B_H(v_{i-1})\right]\left[B_H(v_l)-B_H(v_{l-1})\right]E\left[B_H(v_j)-B_H(v_{j-1})\right]\left[B_H(v_k)-B_H(v_{k-1})\right]\Big\}.$$

There are three similar terms on the right-hand side. We consider the first. The others can be treated in a similar way. Let $\Delta v_r = v_r - v_{r-1}$. Then consider for every $v_i,\ v_{i-1}, v_{j-1}$ and $v_j \in \pi_n \cup \pi_m$,

$$\lim_{\Delta v_i,\ \Delta v_j \to 0} E\left\{\frac{\left[B_H(v_i)-B_H(v_{i-1})\right]}{\Delta v_i}\frac{\left[B_H(v_j)-B_H(v_{j-1})\right]}{\Delta v_j}\right\} \to V_H H(2H-1)\left|v_i - v_j\right|^{2H-2}$$

from Barton and Poor (1988, eqs. 2.5 & 2.13) and using Lemma 5.2. This expression is also bounded for $v_i,\ v_j \in [0,\ t]$. Hence for a sufficiently refined partition $\pi_n \cup \pi_m$, there must be an $\epsilon > 0$ independent of v_i and v_j such that,

$$V_H H(2H-1)|v_i - v_j|^{2H-2} - \epsilon \leq$$

$$E\left\{\frac{\left[B_H(v_i)-B_H(v_{i-1})\right]}{\Delta v_i}\frac{\left[B_H(v_j)-B_H(v_{j-1})\right]}{\Delta v_j}\right\} \leq$$

$$\mathrm{V}_H H(2H-1)|v_i - v_j|^{2H-2} + \epsilon.$$

Thus we have for a sufficiently refined partition $\pi_{n\cup}\pi_m$,

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{k}\zeta_n(v_{i-1}, v_{j-1})\zeta_n(v_{k-1}, v_{l-1})$$

$$\left[V_H H(2H-1)|v_i - v_j|^{2H-2} - \epsilon\right]\left[V_H H(2H-1)|v_k - v_l|^{2H-2} - \epsilon\right]\Delta v_i \Delta v_j \Delta v_k \Delta v_l \leq$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{k}\zeta_n(v_{i-1}, v_{j-1})\zeta_n(v_{k-1}, v_{l-1})$$

$$E\left[B_H(v_i)-B_H(v_{i-1})\right]\left[B_H(v_j)-B_H(v_{j-1})\right]E\left[B_H(v_k)-B_H(v_{k-1})\right]\left[B_H(v_l)-B_H(v_{l-1})\right] \leq$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{k}\zeta_n(v_{i-1}, v_{j-1})\zeta_n(v_{k-1}, v_{l-1})$$

$$\left[V_H H(2H-1)|v_i - v_j|^{2H-2} + \epsilon\right]\left[V_H H(2H-1)|v_k - v_l|^{2H-2} + \epsilon\right]\Delta v_i \Delta v_j \Delta v_k \Delta v_l. \qquad (5.5)$$

Taking $\lim\inf$ across the left-hand inequality in (5.5) and $\lim\sup$ across the right-hand inequality and since the end expressions are Riemann integrable, we obtain

$$\int_0^t\int_0^s\int_0^t\int_0^r \zeta(s,\tau)\zeta(r,\ \alpha)[V_H H(2H-1)|s-\tau|^{2H-2} - \epsilon][V_H H(2H-1)|r-\alpha|^{2H-2} - \epsilon]\,ds\,d\tau\,dr\,d\alpha \leq$$

$$\liminf \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{n} \sum_{l=1}^{k} \zeta_n(v_{i-1}, v_{j-1})\zeta_n(v_{k-1}, v_{l-1})$$

$$E\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big]E\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big] \leq$$

$$\limsup \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{n} \sum_{l=1}^{k} \zeta_n(v_{i-1}, v_{j-1})\zeta_n(v_{k-1}, v_{l-1})$$

$$E\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big]E\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big] \leq$$

$$\int_0^t\int_0^s\int_0^t\int_0^r \zeta(s,\tau)\zeta(r,\,\alpha)[V_H H(2H-1)|s-\tau|^{2H-2} + \epsilon][V_H H(2H-1)|r-\alpha|^{2H-2} + \epsilon]ds d\tau dr d\alpha.$$

But $\epsilon > 0$ was arbitrary so that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{n} \sum_{l=1}^{k} \zeta_n(v_{i-1}, v_{j-1})\zeta_n(v_{k-1}, v_{l-1})$$

$$E\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big]E\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big] =$$

$$\int_0^t\int_0^s\int_0^t\int_0^r \zeta(s,\tau)\zeta(r,\,\alpha)[V_H H(2H-1)|s-\tau|^{2H-2}][V_H H(2H-1)|r-\alpha|^{2H-2}]ds d\tau dr d\alpha. \qquad (5.6)$$

Let us denote the integral in (5.6) by C for convenience. It follows immediately that

$$\lim_{n,m\to\infty} E\Big|\sum_{i=1}^{n} \sum_{j=1}^{i} \zeta_n(v_{i-1}, v_{j-1})\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big] - \sum_{k=1}^{m} \sum_{l=1}^{k} \zeta_m(v_{k-1}, v_{l-1})\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big]\Big|^2 < \infty.$$

We expand this expression

$$E\Big|\sum_{i=1}^{n} \sum_{j=1}^{i} \zeta_n(v_{i-1}, v_{j-1})\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big] - \sum_{k=1}^{m} \sum_{l=1}^{k} \zeta_m(v_{k-1}, v_{l-1})\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big]\Big|^2 =$$

$$E\Big[\sum_{i=1}^{n} \sum_{j=1}^{i} \zeta_n(v_{i-1}, v_{j-1})\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big]\Big]^2 -$$

$$2E\Big[\sum_{i=1}^{n} \sum_{j=1}^{i} \zeta_n(v_{i-1}, v_{j-1})\big[B_H(v_i) - B_H(v_{i-1})\big]\big[B_H(v_j) - B_H(v_{j-1})\big] \sum_{k=1}^{m} \sum_{l=1}^{k} \zeta_m(v_{k-1}, v_{l-1})\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big]\Big] +$$

$$E\Big[\sum_{k=1}^{m} \sum_{l=1}^{k} \zeta_m(v_{k-1}, v_{l-1})\big[B_H(v_k) - B_H(v_{k-1})\big]\big[B_H(v_l) - B_H(v_{l-1})\big]\Big]^2.$$

Taking limits as n and m go to ∞, we have

$$\lim_{n,m\to\infty} E\Big| \sum_{i=1}^{n} \sum_{j=1}^{i} \zeta_n(v_{i-1}, v_{j-1}) \left[B_H(v_i) - B_H(v_{i-1})\right] \left[B_H(v_j) - B_H(v_{j-1})\right] - \sum_{k=1}^{m} \sum_{l=1}^{k} \zeta_m(v_{k-1}, v_{l-1}) \left[B_H(v_k) - B_H(v_{k-1})\right] \left[B_H(v_l) - B_H(v_{l-1})\right] \Big|^2 =$$

$$C - 2C + C = 0. \qquad \blacksquare$$

Definition: The stochastic integral $\int_0^t X(s)dB_H(s)$ is defined as the quadratic mean limit of $\sum_{j=1}^{n} \sum_{k=1}^{j} \zeta_n(v_{j-1}, v_{k-1})[B_H(v_k) - B_H(v_{k-1})]\ [B_H(v_j) - B_H(v_{j-1})]$ as $n \to \infty$.

Theorem 5.4: The stochastic integral $\int_0^t X(s)dB_H(s)$ exists and is well defined. The usual properties of an integral hold.

Proof:

By Lemma 5.3, $\sum_{j=1}^{n} \sum_{k=1}^{j} \zeta_n(v_{j-1}, v_{k-1})[B_H(v_k) - B_H(v_{k-1})]\ \ [B_H(v_j) - B_H(v_{j-1})]$ is a Cauchy sequence. Thus, we have $\sum_{j=1}^{n} \sum_{k=1}^{j} \zeta_n(v_{j-1}, v_{k-1})[B_H(v_k) - B_H(v_{k-1})]$ converges in quadratic mean to a limit process since the space is complete. It is straightforward to show that the ordinary properties of an integral hold since the integral is approximated by the double sum. $\blacksquare$

Definition: The stochastic integral, $\int_0^t X(s)dX(s)$, is defined as $\int_0^t \theta(s)\left[X(s)\right]^2 ds + \int_0^t X(s)dB_H(s)$.

6. Summary

We have considered the stochastic differential equations, $dX(t) = \theta X(t)dt + dB_H(t);\ \ t > 0,$ and $dX(t) = \theta(t)X(t)dt\ \ + dB_H(t);\ \ t > 0$ where $B_H(t)$ is fractional Brownian motion. We have found solutions for these differential equations and have shown the existence of the integrals related to these solutions. We then showed that $B_H(t)$ is not a martingale. This implies that several conventional methods for defining integrals on fractional Brownian motion are inadequate. We formally demonstrated the existence of an estimator for θ or $\theta(t)$ but that estimator depended on the existence of integrals which we did not know existence. We concluded by showing the existence and Riemann sum approximations for these integrals.

References

Barton, R. J. and Poor, V. H. (1988), "Signal detection in fractional Gaussian noise," ***IEEE Transactions on Information Theory***, 34, 943-959

Christopeit, N. (1986), "Quasi-least-squares estimation in semimartingale regression models," ***Stochastics***, 16, 255-278

Cramer, H. and Leadbetter, M. R. (1967), ***Stationary and Related Stochastic Processes***, John Wiley and Sons, Inc.: New York

Dobrushin, R. (1979), "Gaussian and their subordinated generalized fields," ***Annals of Probability***, 7, 1-28

Gregotski, M. E., Jensen, O., and Arkani-Hamed, J. (1991), "Fractal stochastic modeling of aeromagnetic data," ***Geophysics***, 56(11), 1706-1715

Major, P. (1981), ***Multiple Wiener-Ito Integrals***, Lecture Notes in Mathematics, Springer-Verlag: New York

Mandelbrot, B. B. (1983), ***The Fractal Geometry of Nature***, W. H. Freeman and Company: New York

Shiryayev, A. N. (1984), ***Probability***, Springer-Verlag: New York

Soong, T. T. (1973), ***Random Differential Equations in Science and Engineering***, Academic Press, Inc.: New York

Stewart, C. V., Moghaddam, B., Hintz, K. J., and Novak, L. M. (1993), Fractional Brownian motion models for synthetic aperture radar imagery scene segmentation," ***Proceedings of the IEEE***, 81(10), 1511-1522

Wegman, E. J. and Habib, M. K. (1992), "Stochastic methods for neural systems," ***J. Statistical Planning and Inference,*** 33, 5-26